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EFFECT OF PERIODIC BOTTOM ROUGHNESS ON GRAVITATIONAL WAVES IN A LIQUID

V. I. Merkulov, A. P. Mikhailov,
and S. O. Savchenko

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The effect of a rigid bottom of periodic form on small periodic oscillations of the free surface of a liquid is considered with the assumption of low amplitude roughness. The methodologically most significant study in this direction, [1], will be utilized. In [1] the steady-state problem for flow over an arbitrarily rough bottom was studied. Other studies have recently appeared on small free oscillations above a rough bottom. Essentially these have considered the effect of underwater obstacles and cavities on surface waves in the shallow-water approximation (for example, [2], [3]). Liquid oscillations in a layer of arbitrary depth slowly varying with length were considered in [4]. However, these results cannot be applied to the study of resonant interaction of gravitational waves with a periodically curved bottom.

1. We will consider a plane layer of nonviscous incompressible liquid extending infinitely in the x-direction and lying in a gravitational field above a periodically rough bottom (Fig. 1). The form of the bottom is specified by the function $a\lambda(x)$, where a is the amplitude of the bottom roughness, H is the mean height of the layer. The studies were carried out using dimensionless variables, the wavelength of the bottom period being taken equal to 2π. Only periodic small free oscillations of the layer at rest were studied, with the conditions $a \ll 1$, $a/H \ll 1$.

Let φ be the velocity potential. Then if the liquid density is much greater than the density of air and $(\nabla\varphi)^2$ is small, a linearized boundary problem for small free oscillations on a rough bottom is known (see, for example, [5, 6]):

$$\Delta\varphi = 0,\ \frac{\partial\varphi}{\partial \mathbf{n}} = 0\Big|_{y=a\lambda(x)},\ \frac{\partial\varphi}{\partial y} = -\frac{1}{g}\frac{\partial^2\varphi}{\partial t^2}\Big|_{y=H} \tag{1.1}$$

or for a single harmonic $\varphi = e^{i\omega t}u(x, y)$

$$\Delta u = 0,\quad \frac{\partial u}{\partial \mathbf{n}} = 0\Big|_{y=a\lambda(x)},\quad \frac{\partial u}{\partial y} = \frac{\omega^2}{g}u\Big|_{y=H}, \tag{1.2}$$

where $\mathbf{n}$ is a unit vector normal to the line of the bottom.

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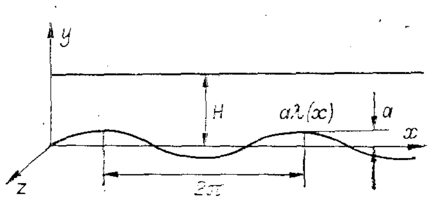

Fig. 1

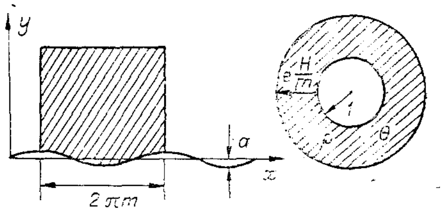

Fig. 2

The surface wave period must be equal to an integral multiple of the bottom roughness period. This can be proven by assuming the opposite, and requiring periodicity of the velocity potential and all its derivatives and continuity of these values along y.

2. Let the period of the surface wave equal m periods of bottom roughness. The region occupied by liquid between maxima of the bottom roughness, located at a spacing of $2\pi m$ from each other along the x-coordinate, can be conformally mapped into a ring so that the bottom line transforms to the outer circumference and the surface line, to the inner circumference, as was done in [1] (Fig. 2).

The liquid-filled region is mapped from the plane z into a region on the plane ξ with polar coordinates ρ, θ. To the accuracy of small terms of second order in a the converse mapping has the form

$$z = -im \ln \xi + af(\xi) + iH,$$

as follows from the results of [7]. For the modulus of the derivative on the inner circumference we find

$$|z'_\xi|_{\rho=1} = m\left(1 + \frac{a}{m}\alpha(\theta)\right) = \chi(\theta),$$

where $\alpha(\theta) = 2(\mathrm{Re}f' + \mathrm{Im}f')|_{\rho=1}$, $\alpha(\theta)$ is a function dependent on the bottom profile and obviously periodic with the period selected. Thus, in place of boundary problem (1.1), (1.2), we obtain a new boundary problem for the bounded area in the plane ξ:

$$\Delta\varphi(\rho, \theta, t) = 0, \ \Delta u(\rho, \theta) = 0; \tag{2.1}$$

$$\frac{\partial\varphi(\rho, \theta, t)}{\partial\rho} = 0\Big|_{\rho=e^{H/m}}, \quad \frac{\partial u(\rho, \theta)}{\partial\rho} = 0\Big|_{\rho=e^{H/m}}; \tag{2.2}$$

$$\frac{\partial\varphi(\rho, \theta, t)}{\partial\rho} = -\frac{1}{g}\chi\frac{\partial^2\varphi}{\partial t^2}\Big|_{\rho=1}, \quad \frac{\partial u}{\partial\rho} = \frac{\omega^2}{g}\chi u\Big|_{\rho=1}, \tag{2.3}$$

where $\varphi(\rho, \theta, t) = e^{i\omega t}u(\rho, \theta)$.

In the future we will consider the second boundary problem for the potential $u(\rho, \theta)$. This is the classical Neumann problem for a ring. With the solution of this problem presented in [8], for the case of equality to zero of the derivative of the velocity potential on the outer circumference at ρ = 1, the solution of boundary problem (2.1)-(2.3) reduces to an integral equation for the unknown natural oscillations of the free surface:

$$u(\theta)|_{\rho=1} = B_0 - \frac{\omega^2}{\pi g}\sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{cth}\left(\frac{kH}{m}\right)(a_k\cos k\theta + b_k\sin k\theta).$$

The series $\sum_k (a_k\cos k\theta + b_k\sin k\theta)$ is the Fourier expansion of the unknown finite velocity potential, so that the sequence of its partial sums is finite, and the sequence $\{\mathrm{cth}(kH/m)/k\}$ decreases monotonically and converges to zero. By Dirichlet's principle such a series converges uniformly. We can interchange the order of integration and summation operations, obtaining the equation

$$u(\theta)|_{\rho=1} = B_0 - \frac{\omega^2}{\pi g}\int_{-\pi}^{\pi} u(\varphi)H(\theta, \varphi)\,d\varphi, \tag{2.4}$$

where

$$H(\theta, \varphi) = \sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{cth}\left(\frac{kH}{m}\right)\chi(\varphi)\cos k(\vartheta - \varphi).$$

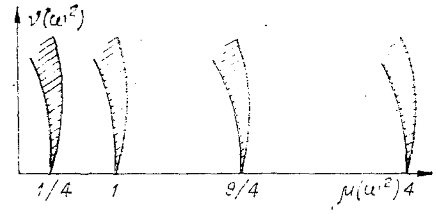

Fig. 3

The series in Eq. (2.4) does not converge at $\theta = \varphi$, but the singularity which appears in this case asymptotically has the form $\ln \sin(\theta - \varphi)$, i.e., is integrable.

We have obtained the most general equation for free surface oscillations in the given formulation. It is important to note that the problem within a region was reduced to a problem on the free surface line, and that the desired function $u(\theta)$ depends only on the one variable θ.

Below we will use $u(\theta)$ to indicate the potential value on the boundary $\rho = 1$.

Let $H \ll 1$ and the first harmonics be the carriers (the maximum symmetry case). Then from Eq. (2.4) we can obtain the Hill equation

$$u_\theta'' + \frac{m^2\omega^2}{gH}\left(1 + \frac{a}{m}\alpha(\theta)\right)u = 0, \tag{2.5}$$

or

$$u_\theta'' + [\mu(\omega^2) + \nu(\omega^2)\alpha(\theta)]u = 0,$$

where

$$\mu(\omega^2) = m^2\omega^2/gH; \; \nu(\omega^2) = am\omega^2/gH.$$

A similar result follows from shallow-water theory (see for example [5]).

Figure 3 shows the stability diagram of this equation presented in [9], with instability regions cross-hatched. These correspond to solutions which increase without limit along coordinate θ and do not relate to the formulation of the problem.

The bases of the instability "wedges" with coordinates $n^2/4$ correspond to resonant frequencies (as understood in [9]).

As was shown in [9], boundaries between stability and instability regions with even numbers correspond to a periodic solution, but in this case the second independent solution of Eq. (2.5) is infinite. If at some point in the parameter plane the boundaries of stability regions contact each other, which corresponds to a double root of the eigenvalue problem, then both independent solutions of Eq. (2.5) are periodic and finite for these parameters. Inasmuch as Eq. (2.5) was obtained to the accuracy of terms of second-order smallness in a, the narrow instability regions (with width of the order of a^2) can be considered approximately constricted into a line, while both independent solutions of Eq. (2.5) are finite and periodic.

Thus periodic oscillations are possible only at frequencies equal to the accuracy of terms second order in a to resonant frequencies with even numbers and corresponding to narrow instability regions.

3. We will consider the case where the bottom profile has the form

$$a\lambda(x) = a \sin x \text{ and } m = 1.$$

Performing the mapping, rectifying a band of unit width [7], and performing certain additional transformations, we obtain an image which rectifies and rotates by $\pi/2$ a slightly curved band of width H:

$$W = z - \frac{i\pi H}{2} + \frac{i}{2}\int_{-\infty}^{\infty} y_0(t)\,\mathrm{cth}\left[\frac{\pi}{2H}(-z-t)\right]dt + \frac{i}{2}\int_{-\infty}^{\infty}(1-y(t))\,\mathrm{th}\left[\frac{\pi}{2H}(-z-t)\right]dt, \tag{3.1}$$

where $1 - y(t)$ specifies the profile of the bottom boundary, and $y_0(t)$ gives the top boundary:

$$1 - y(t) = a \sin t, \; y_0(t) = 0.$$

The latter integral of Eq. (3.1) is calculated, and the final image on the ring has the form

$$\xi = \exp\left[i\left(z - \frac{a}{\operatorname{sh} H}\cos z - \frac{\pi}{2}\right)\right].$$

The modulus of the derivative of the converse image:

$$|z'_\xi| = \left(1 - \frac{a}{\operatorname{sh} H}\cos\theta\right) = \chi(\theta).$$

With consideration of the relationships obtained Eq. (2.5) transforms to Mathieu's equation:

$$u'' + \frac{\omega^2}{gH}\left(1 - \frac{a}{H}\cos\theta\right)u = 0, \quad \frac{a}{H} \ll 1, \quad H \ll 1.$$

We seek a solution by the perturbation method:

$$u^n = u_0^n + \frac{a}{H}u_1^n, \quad \omega_n^2 = (\omega_0^n)^2 + \Delta,$$

$$u_0^n = C\sin n\theta + C'\cos n\theta, \quad (\omega_0^n)^2 = gHn^2, \quad n = 1, 2, \ldots$$

In the first order of smallness in a/H $\Delta = 0$ for any n in accordance with the results of section 2, since for Mathieu's equation all instability regions corresponding to periodic solutions are narrow [9].

For the eigenfunctions, we obtain in the original coordinates:

$$u^n(x) = C\sin(nx - \omega_n t + \varphi_0) + \frac{a}{H}C(R_1\sin((n+1)x - \omega_n t + \varphi_0) +$$

$$+ R_2\sin((n-1)x - \omega_n t + \varphi_0),$$

$$R_1 = \frac{1}{2}n + A, \quad R_2 = \frac{1}{2}n - B, \quad A = -\frac{n^2}{2(2n+1)}, \quad B = \frac{n^2}{2(2n-1)},$$

$$\varphi_0 = \frac{n\pi}{2}, \quad n = 1, 2, \ldots$$

For n = 1 $R_2 = 0$. A similar expression can be obtained for the expansion in cosines with the same arguments.

Thus, each eigenfunction for the oscillations of the surface of a thin liquid layer with bottom sinusoidal in form can be represented in the form of three (at n = 1, two) traveling waves: a carrier, the length of which corresponds to the wavelength of the unperturbed harmonic, and two satellite waves with wave numbers one more and one less (lagging and leading the carrier) with amplitudes of the order of a/H. At n = 1 only the lagging satellite remains.

We will consider the general integral equation (2.4) for a bottom profile of sinusoidal form:

$$u(\theta) = B_0 + \frac{\omega^2}{gH}\left[\int_{-\pi}^{\pi}\sum_{k=1}^{\infty} u(\varphi)\left(1 - \frac{a}{\operatorname{sh} H}\cos\varphi\right)\frac{\cos k(\theta - \varphi)}{k}\operatorname{ctg} kH d\varphi\right].$$

Substituting the Fourier expansion for u(θ), integrating over φ, and equating to zero the sum of the coefficients for each power of the sine and cosine, we obtain an infinite linear homogeneous system of algebraic equations with matrix of tridiagonal form for the coefficients of the Fourier expansion of the velocity potential:

$$\begin{gathered}(1 - a\omega^2\gamma_{11})C_1 + a\gamma_{12}\omega^2 C_2 = 0,\\ \omega^2 a\gamma_{21}C_1 + (1 - a\omega^2\gamma_{22})C_2 + a\omega^2\gamma_{23}C_3 = 0,\\ \cdots\cdots\cdots\cdots\\ a\omega^2\gamma_{i,i-1}C_{i-1} + (1 - a\omega^2\gamma_{i,i})C_{i,i} + a\omega^2\gamma_{i,i+1}C_{i+1} = 0,\\ \gamma_{i,i} = \frac{1}{g}\frac{\operatorname{cth} iH}{i}, \quad \gamma_{i,i-1} = \gamma_{i,i+1} = \frac{1}{2}.\end{gathered} \tag{3.2}$$

We denote by $\Delta(\omega^2)$ the determinant of this infinite system. An equivalent system can be obtained for determination of C'. The free term is found from the condition

$$\overline{\frac{\partial u}{\partial \rho}} = 0 = \frac{\omega^2}{g}\overline{\chi(\theta)u(\theta)} = \frac{\omega^2}{g}\overline{(1 - \cos\theta)\left[B_0 + \sum_n (C_n\sin n\theta + C'_n\cos n\theta)\right]},$$

$$B_0 = -\frac{1}{2}C_1.$$

TABLE 1

k	$(\omega_1^2)^{(k)}$	$(\omega_2^2)^{(k)}$	$(\omega_3^2)^{(k)}$	$(\omega_4^2)^{(k)}$	$(\omega_5^2)^{(k)}$	$(\omega_6^2)^{(k)}$
4	0,995843	3,952574	8,749723	15,287594		
5	0,995843	3,952538	8,7496662	15,214654	23,274828	
6	0,995843	3,952538	8,74962	15,214345	23,129334	32,507671
...	...	...	...	...	...	...
30	0,995843	3,952538	8,749662	15,214345	23,128842	32,250240
*	0,996799	3,947506	8,739378	15,197952	23,105858	32,222230

	C_1	C_2	C_3	C_4	C_5
φ_1	1,0000	$1,6871\cdot10^{-2}$	$-1,0867\cdot10^{-4}$	$3,8252\cdot10^{-7}$	$8,6579\cdot10^{-10}$
φ_2	$6,6626\cdot10^{-2}$	1,000	$-4,1386\cdot10^{-2}$	$7,2996\cdot10^{-4}$	$7,5742\cdot10^{-6}$
φ_3	$5,1231\cdot10^{-3}$	$9,0943\cdot10^{-2}$	1,0000	$-6,8293\cdot10^{-2}$	$-2,0916\cdot10^{-3}$
...	...	...	...	...	...
	C_8	C_9	C_{10}	C_{11}	C_{12}
φ_{10}	$-5,1921\cdot10^{-2}$	$-3,0899\cdot10^{-1}$	1,0000	$-3,4956\cdot10^{-1}$	$-5,9042\cdot10^{-2}$

TABLE 2

k	$(\omega_1^2)^{(k)}$	$(\omega_2^2)^{(k)}$	$(\omega_3^2)^{(k)}$	$(\omega_4^2)^{(k)}$	$(\omega_5^2)^{(k)}$
3	4,6028	15,2534	27,7351		
4	4,6028	15,2522	27,1445	39,7733	
5	4,6028	15,2520	27,1297	38,5573	51,3917
...	...	...	...	...	...
30	4,6028	15,2520	27,1296	38,4915	49,9251
*	4,6212	15,2319	27,1544	38,5611	49,9330

	C_1	C_2	C_3	C_4	C_5
φ_1	1,00000	$-4,2068\cdot10^{-2}$	$8,5223\cdot10^{-4}$	$1,1815\cdot10^{-5}$	$-1,2843\cdot10^{-7}$
φ_2	$1,3596\cdot10^{-1}$	1,0000	$-1,2815\cdot10^{-1}$	$-8,5989\cdot10^{-3}$	$4,0706\cdot10^{-4}$
φ_3	$2,4793\cdot10^{-2}$	$2,1437\cdot10^{-1}$	1,0000	$2,4944\cdot10^{-1}$	$-3,2501\cdot10^{-2}$
...	...	...	...	...	...
φ_{10}	C_8	C_9	C_{10}	C_{11}	C_{12}
φ_{10}	$-3,3481\cdot10^{-1}$	$-5,8336$	1,0000	$-3,6057$	6,1495
	C_{13}	C_{14}			
φ_{10}	2,5445	$2,1414\cdot10^{-1}$			

We take $C_1=1$ and write system (3.2) in the form of Eq. (3.3) AC = f, with $A_{1i} = 0$ $(i \neq 1)$. Let A be a nondegenerate matrix. Then the corresponding homogeneous system does not have a nonzero solution. Consequently [1], Eq. (3.3) has a uniquely defined solution which satisfies the condition $\sum_i C_i^2 < \infty$. This means that Eq. (3.2) has a unique, to the accuracy of a single arbitrary constant, nonzero solution. We will seek a solution of Eq. (3.3) by the reduction method, which assumes for each approximation k the fulfillment of the condition $\Delta^k(\omega^2) = 0$. By definition $\Delta(\omega^2) = \lim_{k\to\infty} \Delta^k(\omega^2)$, $\Delta^k(\omega_k^2) = 0$ (ω_k will be found from this equation). It is then obvious that if $\lim_{k\to\infty} \omega_k^2 = \omega_\infty^2$, then $\Delta(\omega_\infty^2) = 0$, which ensures finding a unique solution (to the accuracy of an arbitrary constant) of Eq. (3.2) as the limit of the sequence $\{C_n^{(k)}\}$.

System (3.3) was solved numerically, reducing it to the problem of the eigenvalues of the equation $BC = \lambda C$, where λ is related to ω by the expression $\lambda(\omega^2 - 1) = 1$.

For each eigenvalue of Eq. (3.3) the coefficients of the Fourier expansion of the eigenfunctions can be found.

The calculation results are presented in Tables 1 and 2 for H = 0.1, a = 0.01; H = 0.5, a = 0.1; $\varphi_1, \ldots, \varphi_n$ are the largest coefficients of the eigenfunction Fourier transforms; $(\omega_n^2)^{(k)}$ is the k-th approximation of the eigenfrequency.

Analysis of Tables 1 and 2 reveals the rapid convergence of the successive approximations for the eigenvalues. For the first eigenvalues there is a "triadicity" of the Fourier expansions, which confirms the results obtained by the perturbation method for Mathieu's equation.

The squares of the frequencies in the table row denoted by * correspond to the case a = 0.

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